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Explicit solution to dynamic portfolio choice problem :

The continuous-time detour

François Legendre, Djibril Togola

UNIVERSITÉ PARIS-EST

Explicit solution to dynamic portfolio choice problem: the continuous-time detour

François Legendre* Djibril Togola†

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Abstract

This paper solves the dynamic portfolio choice problem. Using an explicit solution with a power utility, we construct a bridge between a continuous and discrete VAR model to assess portfolio sensitivities. We find, from a well analyzed example that the optimal allocation to stocks is particularly sensitive to Sharpe ratio. Our quantitative analysis highlights that this sensitivity increases when the risk aversion decreases and/or when the time horizon increases. This finding explains the low accuracy of discrete numerical methods especially along the tails of the unconditional distribution of the state variable.

Keywords: Dynamic portfolio choice; Long-term investing; Time aggregation; Explicit solution; Numerical solution.

JEL Classification: G11; G12.

Introduction

Since at least Merton (1971), many results on portfolio optimization problems have been obtained in a continuous time framework. It is still difficult to solve optimal portfolio problems when there is some degree of predictability in asset returns, *i.e.* when the investment opportunities are time-varying. A great number of papers have proposed to use a VAR model to forecast returns and study its implication for the long-term portfolio choice problem. As a result the academic literature has followed two main directions. The first one relies on mathematical tools and establishes some explicit solutions (see among others Kim and Omberg (1996), Liu (2007) and references therein). The second line of research consists to implement some challenging numerical methods. In fact, Barberis (2000) developed a discretization state space method which serves as a benchmark. Brandt et al. (2005), van Binsbergen and Brandt (2007), Garlappi and Skoulakis (2009) among others use some sophisticated backward induction techniques and evaluate the accuracy of their results by comparing them to the discretization state space benchmark. However, all discrete numerical procedures approximate directly or indirectly a highly non linear value function and cannot explicitly separate the so-called *hedging demand* from the so-called *myopic demand*. Garlappi and Skoulakis (2011) provide a general discussion about approximations accuracy in discrete time.

*Érudite, Université Paris-Est, and TEPP, `F.Legendre@u-pec.fr`.

†✉ Érudite, Université Paris-Est, `Djibril.Togola@u-pec.fr`.

This paper works in continuous time and uses the explicit solution for the portfolio choice problem, then constructs a bridge between continuous and discrete VAR model as in Campbell et al. (2004). In fact, these authors provided evidence that there should exist minor discrepancies between results under discrete and continuous time models. Thus, numerical results that we derive from continuous time are indirectly comparable to those of Garlappi and Skoulakis (2009). We show that, for large degrees of risk aversion and/or small horizon, when the state variable closes to its unconditional mean, the two numerical results are quite similar. Otherwise, results under our explicit solution in continuous time exhibit some discrepancies with Garlappi and Skoulakis (2009) when the risk aversion decreases and/or when the time horizon increases. We argue that this is due to large sensitivity of total demand to state variable (Sharpe ratio).

The paper is organized as follows. Section 1 exposes the way we map the continuous-time investment opportunity set and the discrete-time one. Section 2 gives some insights on the explicit solution for the long-term investor with CRRA preferences. Section 3 gives some numerical results based on Brandt et al. (2005) example.

1 Investment opportunity sets

We first expose the model in a continuous-time framework and in a discrete-time framework to study the impact of a predictable component in stock returns. Next, we show how to recover continuous-time parameters that are consistent with discrete-time VAR estimates.

1.1 Opportunity set in continuous time

We start by assuming that two assets are available for the investor (Campbell et al. (2004) and Kim and Omberg (1996) among others). On the one hand, the riskless asset pays back a constant interest rate r

$$\frac{\mathrm{d}P_t^f}{P_t^f} = r\ \mathrm{d}t \tag{1}$$

where P_t^f denotes the price of this asset at time t. On the other hand, there is a risky asset whose price P_t satisfies the following diffusion process

$$\frac{\mathrm{d}P_t}{P_t} = \mu_t\ \mathrm{d}t + \sigma\ \mathrm{d}B_t^p \tag{2}$$

where B_t^p denotes a scalar Brownian motion with zero drift and unit variance rate. The drift rate μ_t follows a diffusion process as well. It is supposed to be time-varying and state variable dependent. The volatility of the risky asset is assumed to be constant. This is not a strong assumption for the long-term investor (see Campbell and Viceira (2002)). Let X_t denote the Sharpe ratio i.e. the market price of risk/reward for buying/selling one unit of risky asset

$$X_t = \frac{\mu_t - r}{\sigma} \tag{3}$$

Then the Sharpe ratio is assumed to follow the usual "Ornstein-Uhenbeck" diffusion process

$$\mathrm{d}X_t = \kappa(\theta - X_t)\ \mathrm{d}t + \zeta\ \mathrm{d}B_t^x \quad \kappa, \theta, \zeta > 0 \tag{4}$$

where B_t^x denotes another scalar Brownian motion with zero drift and unit variance rate. Parameters θ and κ denote respectively the unconditional mean and the mean reverting parameter of the Sharpe ratio X_t. In fact, parameter κ reflects the rate by which the shocks on Sharpe ratio dissipate and then reverts towards its long-term mean θ. Finally, parameter ζ denotes the instantaneous volatility of Sharpe ratio. It controls the diffusion rate of the process.

The above equations imply that instantaneous return on stocks $\mathrm{d}P_t/P_t$ follows a diffusion process whose drift is mean-reverting and whose innovations are correlated with those of the market price of risk itself, with the correlation coefficient ρ. Thus the following equations hold.

$$\mathrm{d}P_t/P_t = (\sigma X_t + r)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t^p \tag{5}$$

$$\mathrm{d}X_t = \kappa(\theta - X_t)\,\mathrm{d}t + \zeta\,\mathrm{d}B_t^x \tag{6}$$

with $\mathrm{d}B_t^p\,\mathrm{d}B_t^x = \rho\,\mathrm{d}t$. Equations (5) and (6) define a joint stochastic process in continuous time.

1.2 Opportunity set in discrete time

The standard model in discrete time is a restricted VAR(1) process which captures predictability of stocks returns (see Barberis (2000) for instance). We focus on the example analyzed in Brandt et al. (2005) that was reused in van Binsbergen and Brandt (2007) and in Garlappi and Skoulakis (2009). The log excess returns of the risky asset $\Delta\ln P_{t+1} - r^f$ are assumed to be predictable by the log dividend-to-price ratio z_t (r^f denotes the risk-free rate and is equal to 6% in annualized basis). The joint dynamics of these two variables are specified such that

$$\Delta\ln P_{t+1} - r^f = a_r + b_r z_t + \varepsilon_{t+1}^r \tag{7}$$

$$z_{t+1} = a_z + b_z z_t + \varepsilon_{t+1}^z \tag{8}$$

with

$$\begin{pmatrix} \varepsilon_{t+1}^r \\ \varepsilon_{t+1}^z \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_r^2 & \sigma_{rz} \\ \sigma_{rz} & \sigma_z^2 \end{pmatrix}\right] \tag{9}$$

In fact, Campbell and Shiller (1988) forcefully claim that, if returns are predictable, at least, the log dividend-to-price ratio should capture some of that predictability. A substantial long-standing empirical literature has documented many properties of these two regressions. Brandt et al. (2005) report the following estimated values (using the CRSP U.S. quarterly index from January 1986 to December 1995)

$$b_r = 0.060 \quad b_z = 0.958 \quad \frac{\sigma_{rz}}{\sigma_r\,\sigma_z} = -0.941$$

The returns are weakly predictable, the dividend yield is highly persistent and the shocks are strongly negatively related.

1.3 Recovering continuous-time parameters from discrete-time VAR

We closely follow Campbell et al. (2004) to recover the parameters of the continuous-time system eqs (5)–(6) from the restricted VAR(1) eqs (7)–(8). However, Campbell et al. (2004) use the risk premium as state variable; we prefer to use the Sharpe ratio. In matrix form, the discrete-time VAR eqs (7)–(8) is

$$\begin{pmatrix} \Delta\ln P_{t+1} - r^f \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} a_r \\ a_z \end{pmatrix} + \begin{pmatrix} 0 & b_r \\ 0 & b_z \end{pmatrix}\begin{pmatrix} \Delta\ln P_t - r^f \\ z_t \end{pmatrix} + \begin{pmatrix} \varepsilon_{t+1}^r \\ \varepsilon_{t+1}^z \end{pmatrix} \tag{10}$$

The first step is to aggregate the continuous-time model over a span of time taking point observations at evenly spaced points $\{t_0, t_1, \ldots, t_n, t_{n+1}, \ldots\}$, with $\Delta t = t_n - t_{n-1}$. We then obtain, using the discretization method developed by Bergstrom (1984)

$$\begin{aligned}\begin{pmatrix} \Delta\ln P_{t_n+\Delta t} - r\Delta t \\ X_{t_n+\Delta t} \end{pmatrix} = &\begin{pmatrix} (-\sigma^2/2 + \sigma\theta)\Delta t - (1 - \mathrm{e}^{-\kappa\Delta t})\frac{\sigma\theta}{\kappa} \\ (1 - \mathrm{e}^{-\kappa\Delta t})\theta \end{pmatrix} + \\ &\begin{pmatrix} 1 & (1 - \mathrm{e}^{-\kappa\Delta t})\frac{\sigma}{\kappa} \\ 0 & \mathrm{e}^{-\kappa\Delta t} \end{pmatrix}\begin{pmatrix} \Delta\ln P_{t_n} - r \\ X_{t_n} \end{pmatrix} + \begin{pmatrix} U_{t_n+\Delta t}^p \\ U_{t_n+\Delta t}^x \end{pmatrix}\end{aligned} \tag{11}$$

Discrete-time world		Continuous-time world	
Models			
Brandt et al. (2005)		Kim and Omberg (1996)	
$\Delta \ln P_{t+1} - r^f = a_r + b_r z_t + \varepsilon^r_{t+1}$		$\mathrm{d}P^f_t / P^f_t = r \, \mathrm{d}t$	
$z_{t+1} = a_z + b_z z_t + \varepsilon^z_{t+1}$		$\mathrm{d}P_t / P_t = (\sigma X_t + r) \, \mathrm{d}t + \sigma \, \mathrm{d}B^p_t$	
$\mathrm{V}(\varepsilon^r_t) = \sigma^2_r$		$\mathrm{d}X_t = \kappa(\theta - X_t) \, \mathrm{d}t + \zeta \, \mathrm{d}B^x_t$	
$\mathrm{V}(\varepsilon^z_t) = \sigma^2_s$		$\mathrm{d}B^p_t \, \mathrm{d}B^x_t = \rho \, \mathrm{d}t$	
$\mathrm{Cov}(\varepsilon^r_t, \varepsilon^z_t) = \sigma_{rz}$			
Parameter values			
Brandt et al. (2005)		Our computations eqs (13)–(18)	
r^f	0.015		
a_r	0.227	r	0.015
b_r	0.060	θ	0.111
a_z	−0.155	κ	0.0429
b_z	0.958	σ	0.0060
σ^2_r	0.0060	ζ	0.0542
σ^2_z	0.0049	ρ	−0.941
σ_{rz}	−0.0051		

Table 1: Recovering continuous-time parameters

where

$$\begin{pmatrix} U^p_{t_n+\Delta t} \\ U^x_{t_n+\Delta t} \end{pmatrix} = \int_{\tau=0}^{\Delta t} \begin{pmatrix} 1 & (1-\mathrm{e}^{-\kappa \Delta t})\frac{\sigma}{\kappa} \\ 0 & \mathrm{e}^{-\kappa \Delta t} \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ \zeta\rho & \zeta\sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} \mathrm{d}B^p_{t_n+\tau} \\ \mathrm{d}Z^x_{t_n+\tau} \end{pmatrix} \tag{12}$$

with $\mathrm{d}B^x_t = \rho \, \mathrm{d}B^p_t + \sqrt{1-\rho^2} \, \mathrm{d}Z^x_t$ where B^p_t and Z^x_t are two independent brownian terms.

The second step is to apply a linear transformation for the process X_t in (11) so that we can relate the parameters of the transformed system to the parameters of the matrix form (10) of the discrete-time VAR model. Thus, when we normalize time span $\Delta t = 1$, since everything is in quarter, we get (for $b_z, b_r > 0$)

$$r = r^f \tag{13}$$

$$\theta = \frac{a_z b_r}{\sigma_r (1-b_z)} + \frac{a_r + \sigma^2_r/2}{\sigma_r} \tag{14}$$

$$\kappa = -\ln(b_z) \tag{15}$$

$$\sigma = \sigma_r \tag{16}$$

$$\zeta = b_r \frac{\sigma_z}{\sigma_r} \tag{17}$$

$$\rho = \frac{\sigma_{rz}}{\sigma_r \, \sigma_z} \tag{18}$$

The appendix proves these results. Table 1 shows the value of the parameters of the continuous-time equivalent VAR implied by the Brandt et al. (2005) estimates.

2 Portfolio choice problem in continuous time with CRRA preferences

We can now solve the portfolio choice problem of the investor with long-term horizons who faces to the investment opportunity set described in the previous section. We rely on the recent advances in Honda and Kamimura (2011) who use the verification theorem and show that the explicit solution provided under continuous time is in fact an optimal solution especially for risk aversion greater that one.

We consider an investor with initial wealth $W_{t_0} > 0$ who has only two assets (riskless short-term bond and stocks) available for investment. The financial markets are incomplete. Furthermore, the investor can undertake continuous trading, he has no labor income and only cares about terminal wealth W_T where T is the finite planning horizon. The dynamics of price changes are described by (1) and (5)–(6). If α_t is the share of wealth invested in stocks, the instantaneous wealth would be given by

$$\frac{\mathrm{d}W_t}{W_t} = \alpha_t \frac{\mathrm{d}P_t}{P_t} + (1-\alpha_t)\frac{\mathrm{d}P_t^f}{P_t^f} \tag{19}$$

Properly substituting the dynamics of $\mathrm{d}P_t/P_t$ and $\mathrm{d}P_t^f/P_t^f$, wealth dynamics (also called the budget constraint) becomes:

$$\mathrm{d}W_t = (\alpha_t \sigma X_t + r)W_t \ \mathrm{d}t + \alpha_t \sigma W_t \ \mathrm{d}B_t^p \tag{20}$$

Notice that wealth process reflects uncertainty in instantaneous returns (term $\mathrm{d}B_t^p$) and about the state variable (the term X_t). Given this formalization about wealth process, at time t_0, the investor's optimization problem can then be expressed as

$$\max_{\alpha_{t_0}} \quad \mathrm{E}_{t_0}\, \mathrm{e}^{-\beta T} u(W_T) \quad \text{subject to the constraint (20)} \quad W_{t_0} \text{ fixed} \tag{21}$$

where E_{t_0} denotes the operator of conditional rational expectation at date t_0, β the time discount parameter (with $\beta > 0$) and $u(\cdot)$ the utility function defined over terminal wealth. Let $J(W_{t_0}, X_{t_0}, t_0)$ defines the value of the problem defined in (22) at time t_0

$$J(W_{t_0}, X_{t_0}, t_0) = \max_{\alpha_{t_0}} \quad \mathrm{E}_{t_0}\, \mathrm{e}^{-\beta T} u(W_T) \tag{22}$$

The Bellman equation generalizes this problem for every time t so that

$$J(W_t, X_t, t) = \max_{\alpha_t} \ \mathrm{E}_t\, J(W_t + \mathrm{d}W_t, X_t + \mathrm{d}X_t, t + \mathrm{d}t) \tag{23}$$

Equation (23) emphasizes the fact that current optimal decisions depend on the conditional expected value of the problem which, in turn, is intimately linked to future wealth and the state variable. Applying Ito's lemma to the Bellman equation, we find that

$$\begin{aligned} 0 = \max_{\alpha_t} \Bigg[& \frac{\partial J}{\partial W_t}(\alpha_t \sigma X_t + r)W_t + \frac{\partial J}{\partial t} + \frac{\partial J}{\partial X_t}\kappa(\theta - X_t) + \\ & \frac{1}{2}\frac{\partial^2 J}{\partial^2 W_t}\sigma^2 \alpha_t^2 W_t^2 + \frac{1}{2}\frac{\partial^2 J}{\partial^2 X_t}\zeta^2 + \frac{\partial^2 J}{\partial W_t \partial X_t}\sigma \alpha_t \zeta \rho W_t \Bigg] \end{aligned} \tag{24}$$

The first order condition of equation (24) with respect to α_t implies that

$$\alpha_t^\star = \frac{\partial J/\partial W_t}{\partial^2 J/\partial^2 W_t}\frac{1}{W_t}\frac{X_t}{\sigma} + \frac{\partial^2 J/(\partial W_t \partial X_t)}{\partial^2 J/\partial^2 W_t}\frac{1}{W_t}\frac{\zeta}{\sigma}\rho \tag{25}$$

Merton (1971) was the first to propose such additive decomposition between a *myopic demand* (first term) and a *hedging demand* (second term) of the optimal allocation to stocks. There is no hedging demand especially when the opportunity set is nonstochastic ($\zeta = 0$) or when the opportunity set is uncorrelated with asset returns ($\rho = 0$).

Now, we need to explicitly define the function $J(\cdot)$. The first conjecture (see Kim and Omberg (1996)) is to assume

$$J(W_t, X_t, t) = \mathrm{e}^{-\beta t} u(W_t)\, [f(X_t, t)]^\gamma \tag{26}$$

where $f(\cdot)$ is an auxiliary function with the terminal condition $f(X_T, T) = 1$. We consider the CRRA preferences $u(W_t) = W_t^{1-\gamma}/(1-\gamma)$ where γ is the coefficient of relative risk aversion. Thus, the hedging demand in (25) could straightforward be expressed as

$$\frac{\partial f/\partial X_t}{f}\frac{\zeta}{\sigma}\rho = \frac{\partial \ln f}{\partial X_t}\frac{\zeta}{\sigma}\rho$$

Then, under CRRA preferences hypothesis, the optimal allocation to stocks could be expressed as

$$\alpha_t^\star = \frac{1}{\gamma}\frac{X_t}{\sigma} + \frac{\partial \ln f}{\partial X_t}\frac{\zeta}{\sigma}\rho \tag{27}$$

So, the Bellman equation (24) can be rewritten as

$$\begin{aligned} 0 = & \frac{f_t'}{f} + \frac{1-\gamma}{\gamma}r - \frac{\beta}{\gamma} + \frac{1-\gamma}{2\gamma^2}X_t^2 + \frac{f_x'}{f}\frac{1-\gamma}{\gamma}\zeta X_t\rho + \frac{f_x'}{f}\kappa(\theta - X_t) + \\ & \frac{f_{xx}''}{f}\frac{\zeta^2}{2} + \left(\frac{f_x'}{f}\right)^2\frac{1-\gamma}{2}\zeta^2(\rho^2-1) \end{aligned} \tag{28}$$

where we use intuitive notations for the derivatives of the function $f(\cdot)$. Equation (28) is a partial differential equation which admits analytical solutions especially if utility is logarithmic ($\gamma = 1$ by l'Hopital's rule) or if markets are complete ($\rho = \pm 1$).

The second conjecture is to assume

$$f(X_t, t) = \exp\left[C_0(t) + C_1(t)X_t + \frac{1}{2}C_2(t)X_t^2\right] \tag{29}$$

where $C_0(t)$, $C_1(t)$ and $C_2(t)$ are some undetermined time varying coefficients (with $C_0(T) = C_1(T) = C_2(T) = 0$). Under this conjecture, using equation (27), the optimal allocation to stocks is

$$\alpha_t^\star = \frac{1}{\gamma}\frac{X_t}{\sigma} + [C_1(t) + C_2(t)X_t]\frac{\zeta}{\sigma}\rho \tag{30}$$

We only need to recover $C_1(t)$ and $C_2(t)$ coefficients.

This conjecture was also used by Kim and Omberg (1996) and by Liu (2007) among others. More recently, Honda and Kamimura (2011) show that the explicit solution derived from the Bellman equation is in fact, even if the markets are incomplete, an optimal solution to the problem of the long-term investor who only care about terminal wealth and who have a risk aversion larger than unity.

Let us substitute our second conjecture in the equation (28)

$$\begin{aligned} 0 = & \left[\frac{\mathrm{d}C_2}{\mathrm{d}t} + a\,C_2^2 + b\,C_2 + c\right]X_t^2 + \left[\frac{\mathrm{d}C_1}{\mathrm{d}t} + \frac{b}{2}C_1 + \kappa\theta C_2 + a\,C_1C_2\right]X_t + \\ & \left[\frac{\mathrm{d}C_0}{\mathrm{d}t} + \frac{1-\gamma}{\gamma}r - \frac{\beta}{\gamma} + \kappa\theta C_1 + \frac{\zeta^2}{2}C_2 + \frac{a}{2}C_1^2\right] \end{aligned} \tag{31}$$

where $a = [1 + (1-\gamma)(\rho^2-1)]\,\zeta^2$, $b = 2[(1-\gamma)/\gamma\zeta\rho - \kappa]$ and $c = (1-\gamma)/\gamma^2$. As, whatever the value of X_t, the equation (31) must hold, all terms within brackets are simultaneously set to zero to solve the equation for $C_0(\cdot)$, $C_1(\cdot)$, and $C_2(\cdot)$. Defining the discriminant D

$$D = b^2 - 4\,a\,c$$

one can check that if $\gamma > 1$ then $D > 0$. Thus, the two solutions of interest are given by

$$C_2(t) = \frac{2c\left(1 - \mathrm{e}^{-\delta(T-t)}\right)}{2\delta - (b+\delta)\left(1 - \mathrm{e}^{-\delta(T-t)}\right)} \tag{32}$$

$$C_1(t) = \frac{4c\kappa\theta}{\delta}\frac{\left(1 - \mathrm{e}^{-\delta(T-t)/2}\right)^2}{2\delta - (b+\delta)\left(1 - \mathrm{e}^{-\delta(T-t)}\right)} \tag{33}$$

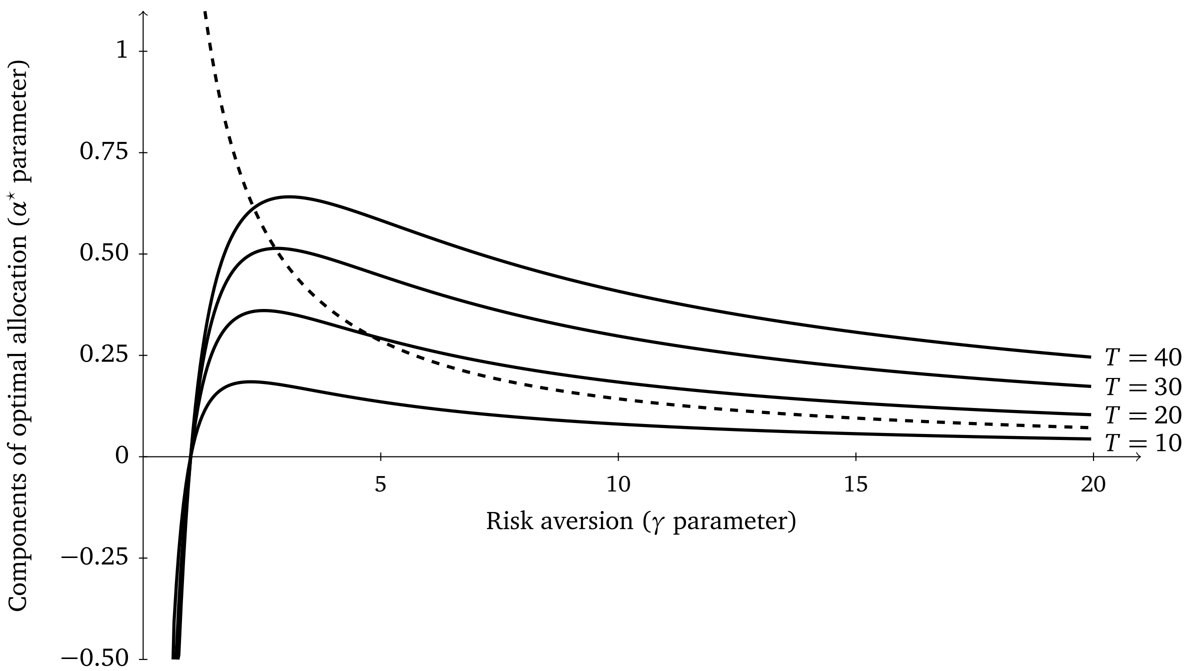


Figure 1: Myopic (dashed line) and hedging (solid line) demands as function of risk aversion for $X_{t_0} = \theta$

where δ denotes $\sqrt{D}$. Kim and Omberg (1996) called this the normal solution and discussed about some alternative solutions those are beyond the scope of this paper. The appendix provides details about (32) and (33). It is easy to see that there is a linear relation between $C_1(\cdot)$ and $C_2(\cdot)$. Then, for $\gamma > 1$, C_1 and C_2 are always negative. As a result, since $\rho < 0$, the hedging demand is always positive when the preferences are not logarithmic (more precisely for $\gamma > 1$) and the market price of risk is positive.

3 Numerical results

As mentioned above, we illustrate our approach using the well documented Brandt et al. (2005) example. Table 1 collects the continuous-time parameters recovered from this example. For comparison purposes, we also use the Garlappi and Skoulakis (2009) results, obtained from the same discrete-time VAR(1) estimates and by means of a sophisticated numerical method.

Figure 1 and table 2 help to understand the long-term investor problem. For $\gamma = 1$ *i.e.* the case of logarithmic utility, no hedging demand is required. For this case, the dynamic portfolio choice reduces to static one whatever the time horizon. Otherwise, for $\gamma > 1$ and horizon longer than one, under CRRA preferences and mean reverting returns, agent should have a positive hedging demand to prevent adverse changes in investment opportunities (Merton, 1971). However, for $\gamma \to \infty$, more specifically for a very conservative agent, stocks are not attractive. Thus, he would not invest into stocks since the total demand (sum of myopic demand and hedging demand) converges toward zero. Our results reset all theses basic important features.

The total demand is sensitive to risk aversion. Results from previous studies imply that myopic and hedging demands are more sensitive to small values of risk aversion. We confirm this and argue that the sensitivity of hedging demand to state variable is maximal near the critical point $\gamma \approx 2$. Our equation (30) and figure 1 show this evidence. To quantitatively see this, just evaluate the derivative of $\alpha^\star$ with respect to X.

Table 2 reports the evidence that both myopic and hedging demand are sensitive to initial value of

T		$\gamma = 5$					$\gamma = 15$				
		$X_{(10)}$	$X_{(30)}$	$X_{(50)}$	$X_{(70)}$	$X_{(90)}$	$X_{(10)}$	$X_{(30)}$	$X_{(50)}$	$X_{(70)}$	$X_{(90)}$
10	**MD**	−34.0	3.0	28.6	54.2	91.1	−11.3	1.0	9.5	18.1	30.4
	HD	−10.9	3.5	13.5	23.6	38.0	−4.6	1.5	5.7	9.9	15.9
20	**MD**	−34.0	3.0	28.6	54.2	91.1	−11.3	1.0	9.5	18.1	30.4
	HD	−15.9	10.8	29.2	47.7	74.3	−7.2	4.9	13.3	21.6	33.7
30	**MD**	−34.0	3.0	28.6	54.2	91.1	−11.3	1.0	9.5	18.1	30.4
	HD	−16.0	19.8	44.7	69.5	105.3	−7.7	9.8	21.9	34.1	51.6
40	**MD**	−34.0	3.0	28.6	54.2	91.1	−11.3	1.0	9.5	18.1	30.4
	HD	−13.2	29.1	58.3	87.6	129.8	−6.5	15.5	30.7	46.0	68.0

For each risk aversion γ, the first line reports the myopic demand (**MD**) and the second line the hedging demand (**HD**) without short selling constraints. We present the results for 5 different initial values of the Sharpe ratio X. Each value corresponds to the p-th percentile of the unconditional distribution of X, defined by equations (49) and denoted by $X_{(p)}$, where p takes values 10, 30, 50, 70, and 90 (then $X_{(50)} = \theta$).

Table 2: Myopic and hedging demands for investment horizon of T quarters (%)

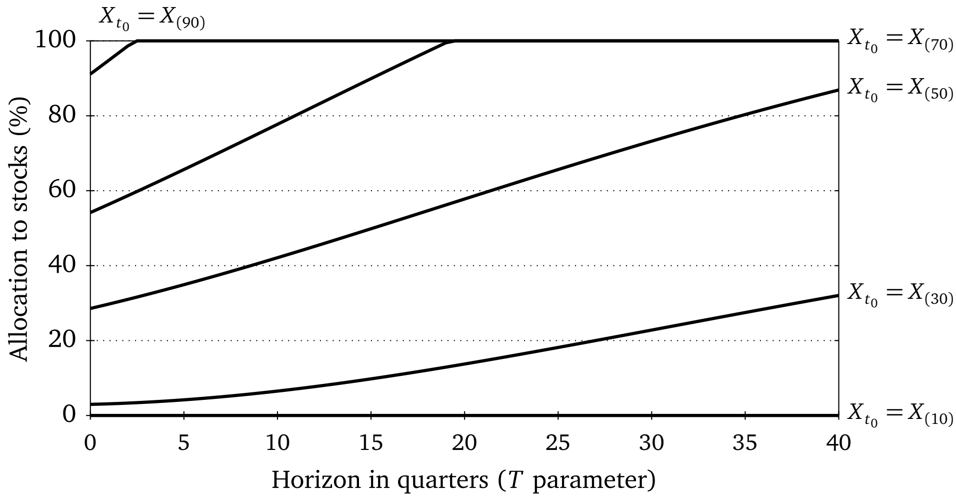
Sharpe ratio. These two components of optimal allocation individually increase with the percentile of the Sharpe ratio unconditional distribution. Thus, the total demand exhibits the same behavior. This is consistent with Campbell et al. (2004) among others. In fact, high Sharpe ratio or equivalently high risk premium relative to volatility signals better investment opportunities. Therefore, optimal fraction to allocate into stocks should increase from the knowledge of mean reverting parameter that serves to quantify the expected Sharpe ratio.

Myopic demand is independent from time horizon while hedging demand increases nonlinearly with time horizon. However, table 2 quantitative figures suggest that this relation is almost linear but small changes in horizon induce substantial hedging demand. Horizon effect is important but quiet monotonic for a given percentile of the state variable unconditional distribution. All changes in total demand for fixed risk aversion and state variable are due to changes in horizon and are large for small risk aversion.

The horizon effect on hedging demand is important in optimal allocation because it widely dominates for longer horizons. In fact, when horizon is greater than 20 quarters, hedging demand becomes always greater than myopic demand when the Sharpe ratio initial value is between 30 and 70 percentiles.

We finally use the common assumption of no-borrowing and short-sale constraints. Thus, in table 3, we restrict all portfolio weights between 0 and 1. One can notice that we generally obtain values fairly close to those of Garlappi and Skoulakis (2009) while frameworks are not the same. Garlappi and Skoulakis (2009) worked in discrete-time and initial values of their state variable are drawn for the unconditional distribution of quarterly dividend price ratio. They use a sophisticated numerical optimization technique. We work in continuous time (no numerical optimization) and our initial values are computed using the unconditional distribution of continuous Sharpe ratio that we discretized in points observation and recovered using the same quarterly dividend price ratio. However, a closer inspection of table 3 figures show that the optimal allocation to stocks is more sensitive to the state variable and to the time horizon than the sensitivity obtained by Garlappi and Skoulakis (2009). We run some numerical simulations, within the discrete-time framework, to evaluate our results in order to find the causes of the discrepancies between the two frameworks. We were unable to qualitatively and quantitatively invalidate our results.

To test our results, we run some forward pure simulations in discrete time. More precisely, for instance, we explore the case where the initial value of the Sharpe ratio is the 30-th percentile ($X_0 = X_{(30)}$, the relative risk aversion is equal to 5 ($\gamma = 5$), and the planning horizon is equal to 10 ($T = 10$



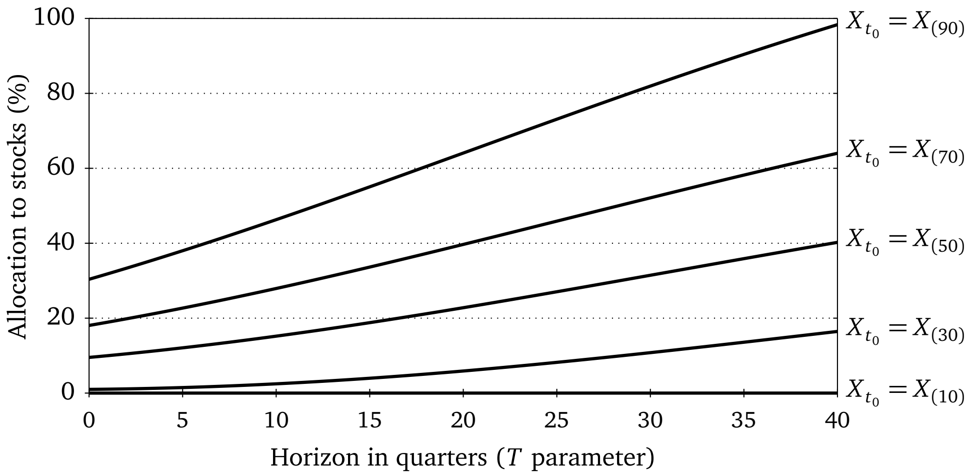


Figure 2: Optimal allocation to stocks as function of the horizon for $\gamma = 5$ (first panel) and for $\gamma = 15$ (second panel) for 5 different initial values of the Sharpe ratio X (as in table 2 or 3)

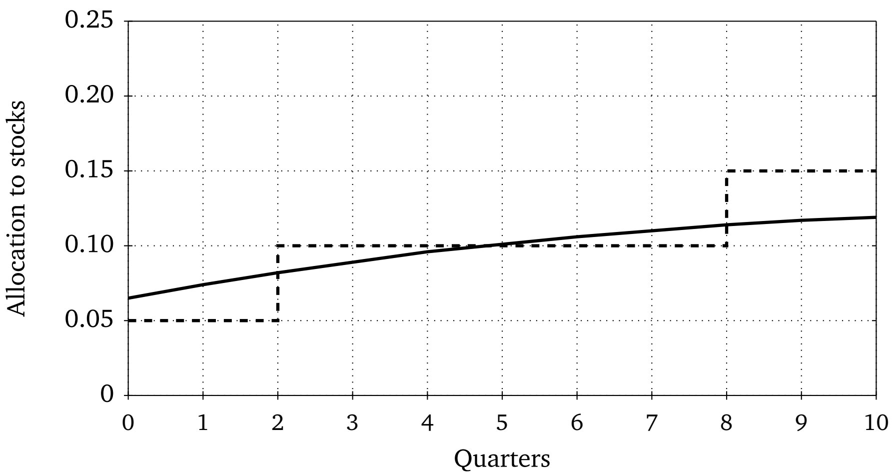


Figure 3: Path of optimal allocation to stocks for $\gamma = 5$, $X_0 = X_{(30)}$, and $T = 10$ obtained by explicit solution (solid line) and by simulation and trial $\{0.05, 0.10, 0.15, 0.20, 0.25\}$ grid (dashed line)

T		$\gamma = 5$					$\gamma = 15$				
		$X_{(10)}$	$X_{(30)}$	$X_{(50)}$	$X_{(70)}$	$X_{(90)}$	$X_{(10)}$	$X_{(30)}$	$X_{(50)}$	$X_{(70)}$	$X_{(90)}$
10	**LT**	0.0	6.5	42.1	77.7	100.0	0.0	2.5	15.2	27.9	46.3
	GS	0.0	13.3	43.2	73.1	100.0	0.0	4.3	15.4	27.0	44.7
	Δ	0.0	−6.8	−1.1	4.6	0.0	0.0	−1.8	−0.2	0.9	1.6
20	**LT**	0.0	13.7	57.8	100.0	100.0	0.0	5.9	22.8	39.7	64.1
	GS	0.0	24.4	57.2	89.7	100.0	0.0	10.7	25.1	40.4	63.2
	Δ	0.0	−10.7	0.6	10.3	0.0	0.0	−4.8	−2.3	−0.7	0.9
30	**LT**	0.0	22.8	73.2	100.0	100.0	0.0	10.8	31.5	52.1	81.9
	GS	0.0	32.8	68.4	100.0	100.0	0.0	17.5	35.2	54.0	80.7
	Δ	0.0	−10.0	4.8	0.0	0.0	0.0	−6.7	−3.7	−1.9	1.2
40	**LT**	0.0	32.0	86.9	100.0	100.0	0.0	16.5	40.2	64.0	98.3
	GS	0.0	38.8	77.6	100.0	100.0	0.0	24.1	44.5	65.7	94.6
	Δ	0.0	−6.8	9.3	0.0	0.0	0.0	−7.6	−4.3	−1.7	3.7

For each risk aversion γ, the first line reports our results (**LT** – optimal allocation to stocks in continuous time), the second line the Garlappi and Skoulakis (2009) results (**GS** – optimal allocation to stocks in discrete time), and the third line reports the difference between our results and Garlappi and Skoulakis (2009) results. We present the results for 5 different initial values of the Sharpe ratio X calibrated using the same estimates involving dividend price ratio as in **GS**. Each value corresponds to the p-th percentile of the unconditional distribution of X, defined by equation (49) and denoted by $X_{(p)}$, where p takes values 10, 30, 50, 70, and 90.

Table 3: Optimal allocation to stocks for investment horizon of T quarters (%)

quarters). With this configuration, when we get an initial optimal allocation to stocks of 0.065, Garlappi and Skoulakis (2009) obtain twice as many (0.133, see table 3). That's large. Thus, we first build a sample of size 100 000 for $z_{t+1}, z_{t+2}, \ldots,$ and z_{t+10} and for $\Delta \ln P_{t+1}, \Delta \ln P_{t+2}, \ldots,$ and $\Delta \ln P_{t+10}$ using the restricted VAR(1) eqs (7)–(8). We choose the grid $G = \{0.05, 0.10, 0.15, 0.20, 0.25\}$ for trial allocations to stocks, to overlay both our and Garlappi and Skoulakis (2009) solutions. Then, for each path in the sample, the value of terminal wealth is computed from the cartesian product $G \times G \times \cdots \times G$ of all possible strategies. The computational burden is very high as we evaluate $5^{10} = 9\,765\,625$ strategies. Figure 3 shows that the forward path in discrete time (no numerical optimization) closes to the path of our explicit solution particularly at the critical starting point, the 30-th percentile of the state variable for small risk aversion ($\gamma = 5$).

Conclusion

We examine the "continuous-time detour" to solve the long-term investor problem when the stock returns are predictable. We obtain an explicit optimal solution in the continuous-time world and, after recovering the continuous-time parameters from the discrete-time world estimates, we reuse such solution to assess the sensitivities of optimal allocation to the initial values of the state variable, to the risk aversion and to the time horizon. We find greater sensitivities than those reported in the literature. We also find that the sensitivity of total demand to the state variable is not uniform along the unconditional distribution of the state variable.

Previous numerical approximation techniques that deal with the problem we consider are subject to some numerical errors. Therefore, they do not always provide accurate results. We show that the hedging demand part of allocation dominates at longer horizons and it is very sensitive to state variable especially when risk aversion decreases and/or the time horizon increases. This finding could explain the low accuracy of discrete numerical methods especially along the tails of the unconditional distribution of the state variable.

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Appendix

A.1 Proof of continuous VAR recovering by discrete VAR

The matrix (10) could be rewritten as

$$\Delta \ln P_{t+\Delta t} - r^f = a_r + b_r z_t + \varepsilon^r_{t+\Delta t} \tag{34}$$

$$z_{t+\Delta t} = a_z + b_z z_t + \varepsilon^z_{t+\Delta t} \tag{35}$$

Equations (34) and (35) describe a joint process of an econometric model in which z denotes the dividend price ratio. The corresponding discretized version of the continuous time model in matrix (11) could be rewritten as

$$\Delta \log P_{t_n+\Delta t} - r\Delta t = (-\sigma^2/2 + \sigma\theta)\Delta t - (1 - \mathrm{e}^{-\kappa\Delta t})\frac{\sigma\theta}{\kappa} + (1 - \mathrm{e}^{-\kappa\Delta t})\frac{\sigma}{\kappa}X_{t_n} + U^p_{t_n+\Delta t} \tag{36}$$

$$X_{t_n+\Delta t} = (1 - \mathrm{e}^{-\kappa\Delta t})\theta + \mathrm{e}^{-\kappa\Delta t}X_{t_n} + U^x_{t_n+\Delta t} \tag{37}$$

Comparing the expectations of (34) and (36), we get

$$z_t = -\frac{a_r}{b_r} + (-\sigma^2/2 + \sigma\theta)\frac{\Delta t}{b_r} - (1 - \mathrm{e}^{-\kappa\Delta t})\frac{\sigma\theta}{b_r\kappa} + (1 - \mathrm{e}^{-\kappa\Delta t})\frac{\sigma}{b_r\kappa}X_{t_n} \tag{38}$$

Iterating forward (38), Δt periods ahead and using (35), we obtain

$$\begin{aligned} -\frac{a_r}{b_r} + (-\sigma^2/2 + \sigma\theta)\frac{\Delta t}{b_r} - (1 - \mathrm{e}^{-\kappa\Delta t})\frac{\sigma\theta}{b_r\kappa} + (1 - \mathrm{e}^{-\kappa\Delta t})\frac{\sigma}{b_r\kappa}X_{t_n+\Delta t} = \\ a_z + b_z\left(-(-\sigma^2/2 + \sigma\theta)\frac{\Delta t}{b_r} + (1 - \mathrm{e}^{-\kappa\Delta t})\frac{\sigma\theta}{b_r\kappa} - (1 - \mathrm{e}^{-\kappa\Delta t})\frac{\sigma}{b_r\kappa}X_{t_n}\right) + \varepsilon^z_{t+\Delta t} \end{aligned} \tag{39}$$

After some algebra, we find that

$$\begin{aligned} X_{t_n+\Delta t} = \left[-\frac{a_z b_r}{\sigma} + (1-b_z)\left(-\frac{a_r}{\sigma} + (-\sigma/2 + \theta)\Delta t - \theta\right)\right]\frac{\kappa}{1 - \mathrm{e}^{-\kappa\Delta t}} + b_z X_{t_n} - \\ \frac{b_r}{\sigma}\frac{\kappa}{1 - \mathrm{e}^{-\kappa\Delta t}}\varepsilon^z_{t+\Delta t} \end{aligned} \tag{40}$$

Notice that $\lim_{\kappa\Delta t\to 0}\left(1 - \mathrm{e}^{-\kappa\Delta t}\right) = \kappa\Delta t$. Finally, comparing equation (40) to (37), equations (13)-(15) directly follow. To compute the associated second moments, one can compute the variance of U vector in (12).

$$\mathrm{Var}\begin{pmatrix} U^p_{t_n+\Delta t} \\ U^x_{t_n+\Delta t} \end{pmatrix} = \int_{\tau=0}^{\Delta t} FF'\begin{pmatrix} \mathrm{d}\tau \\ \mathrm{d}\tau \end{pmatrix} \tag{41}$$

where

$$F = \begin{pmatrix} 1 & (1 - \mathrm{e}^{-\kappa\Delta t})\frac{\sigma}{\kappa} \\ 0 & \mathrm{e}^{-\kappa\Delta t} \end{pmatrix}\begin{pmatrix} \sigma & 0 \\ \zeta\rho & \zeta\sqrt{1-\rho^2} \end{pmatrix} \tag{42}$$

Using a matching procedure involving equations (34)-(41) we can directly reset equations (16)-(18). Furthermore, the resulting general formula for every Δt become such that

$$\mathrm{Var}(X_{t_n+\Delta t}) = \frac{\zeta^2}{2\kappa}(1 - \mathrm{e}^{-2\kappa\Delta t}) \tag{43}$$

$$\mathrm{Cov}(X_{t_n+\Delta t}, \Delta\log P_{t+\Delta t}) = \frac{\rho\sigma\zeta}{\kappa}\left(1 - \mathrm{e}^{-\kappa\Delta t}\right) + \frac{\sigma\zeta^2}{\kappa^2}\left(1 - \mathrm{e}^{-\kappa\Delta t}\right) - \frac{\sigma\zeta^2}{2\kappa^2}\left(1 - \mathrm{e}^{-2\kappa\Delta t}\right) \tag{44}$$

$$\begin{aligned} \mathrm{Var}\left(\Delta\ln(P_{t+\Delta t})\right) = \left(\sigma^2 + 2\rho\frac{\zeta\sigma^2}{\kappa} + \frac{\zeta^2\sigma^2}{\kappa^2}\right)\Delta t - 2\rho\frac{\zeta\sigma^2}{\kappa^2}\left(1 - \mathrm{e}^{-\kappa(\Delta t)}\right) \\ - 2\frac{\zeta^2\sigma^2}{\kappa^3}\left(1 - \mathrm{e}^{-\kappa(\Delta t)}\right) + \frac{\zeta^2\sigma^2}{2\kappa^3}\left(1 - \mathrm{e}^{-2\kappa(\Delta t)}\right) \end{aligned} \tag{45}$$

Where the instantaneous standard deviation of X denoted ζ is given by equation (17). Again, notice that, for small κ, i.e. when $\kappa\Delta t \to 0$, the term $(1 - \mathrm{e}^{-\kappa\Delta t}) \to \kappa\Delta t$. So when $\Delta t = 1$, all second moments could be approximated by their instantaneous counterparts. Otherwise, when $\Delta t \neq 1$, these kinds of approximations become no longer valid. Campbell et al. (2004, p. 2208) discuss about pitfalls resulting for this case. Taking this into account, for instance, one can compute the terminal conditional variances by just setting $\Delta t = T$ and $t = 0$.

The unconditional moments of X that have been used in this paper are derived from equation (38) when Δt is normalized to one.

$$X_{t_n} = \frac{\sigma}{2} + \frac{a_r + b_r z_t}{\sigma} \tag{46}$$

$$\mathrm{E}(X_{t_n}) = \frac{\sigma}{2} + \frac{a_r + b_r \mathrm{E}(z_t)}{\sigma} = \frac{\sigma}{2} + \frac{a_r + b_r a_z/(1-b_z)}{\sigma} \tag{47}$$

Then, the unconditional mean of X is

$$\theta = \frac{a_z b_r}{\sigma_r(1-b_z)} + \frac{a_r + \sigma_r^2/2}{\sigma_r} \tag{48}$$

In fact, we have used the result $\sigma = \sigma_r$ in equation (16) and the fact that z follows an AR(1) process (Brandt et al. (2005) followed by Garlappi and Skoulakis (2009) among others). Thus its unconditional moments are known, $\mathrm{E}\left(z_t\right) = a_z/(1-b_z)$ and $\mathrm{Var}\left(z_t\right) = \sigma_z^2/(1-b_z^2)$ (Hamilton (1994, p. 53)). So, under equation (38), one can match all unconditional percentiles $z_{(p)}$ with their unconditional counterparts $X_{(p)}$ (p denotes the p-th percentile) and get the optimal policies for those values. Since the hedging demand is very sensitive to state variable, we directly draw $X_{(p)}$ from the unconditional distribution of the point observation X_{t_n} of our continuous time state variable X in order to avoid computational errors. As a result, $X_{(50)} = \theta$ and the following unconditional distributions hold.

$$z \sim N\left(\frac{a_z}{1-b_z}, \frac{\sigma_z^2}{1-b_z^2}\right) \implies X \sim N\left(\theta, \frac{b_r^2}{\left(1-b_z^2\right)}\frac{\sigma_z^2}{\sigma_r^2}\right) \tag{49}$$

A.2 Proof of parameters C_1 and C_2

Regarding (31), the solution for C_2 could be derived from the following equation:

$$\frac{\mathrm{d}C_2}{\mathrm{d}t} + a\,C_2^2 + b\,C_2 + c = 0 \tag{50}$$

This equation could straightforward be rewritten as

$$\int_t^T \frac{1}{a\,C_2^2 + b\,C_2 + c}\,\mathrm{d}C_2 = -(T-t) \tag{51}$$

Since parameters a, b and c are constant, given $C_2(T) = 0$, the integral table provides the solution for C_2. Substitute this into the following equation

$$\frac{\mathrm{d}C_1}{\mathrm{d}t} + \kappa\theta C_2 + \left(\frac{b}{2} + a\,C_2\right)C_1 = 0 \tag{52}$$

that we derived from equation (31). Again using the terminal condition $C_1(T) = 0$ and the constant variation method, one can get the solution for C_1 as follows :

$$
\begin{aligned}
C_1(t) &= \int_t^T \mathrm{e}^{-\int_{t+s}^T (b/2+aC_2(u))\,\mathrm{d}u} \left(-\kappa\theta C_2(s)\right)\,\mathrm{d}s \\
&= \int_t^T e^{\left(-b(T-t-s)/2-a\int_{t+s}^T C_2(u)\,\mathrm{d}u\right)} \left(-\kappa\theta C_2(s)\right)\,\mathrm{d}s \\
&= \int_t^T e^{-\delta(T-t-s)/2} \frac{(\delta+b)(\mathrm{e}^{\delta s}-1)+2\delta}{(\delta+b)(\mathrm{e}^{\delta(T-t)}-1)+2\delta} \left(-\kappa\theta C_2(s)\right)\,\mathrm{d}s \\
&= \frac{2c\kappa\theta \mathrm{e}^{\delta(T-t)/2}}{(\delta+b)(\mathrm{e}^{\delta(T-t)}-1)+2\delta} \int_t^T \left(\mathrm{e}^{\delta s/2}-\mathrm{e}^{-\delta s/2}\right)\,\mathrm{d}s \\
&= \frac{4c\kappa\theta/\delta}{-(\delta+b)(1-\mathrm{e}^{\delta(T-t)})+2\delta} \left(\mathrm{e}^{\delta(T-t)/2}-1\right)^2 \\
&= \frac{2\kappa\theta}{\delta} \frac{2c}{2\delta-(\delta+b)(\mathrm{e}^{\delta(T-t)}-1)} \left(1-\mathrm{e}^{\delta(T-t)/2}\right)^2 \\
&= \frac{2\kappa\theta}{\delta} \frac{\left(1-\mathrm{e}^{\delta(T-t)/2}\right)^2}{1-\mathrm{e}^{\delta(T-t)}} C_2(t)
\end{aligned}
$$